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UNIQUENESS OF ASSOCIATION SCHEMES, (U)

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(6) UNIQUENESS OF ASSOCIATION SCHEMES

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1. Graphs, Incidence structures and Association schemes. A graph G is a triple (V , E , I) where V and E are disjoint sets and I is a mapping from E to the subsets of vertices such that for all $e \in E$, I(e) contains at most two elements. Elements of V and E are respectively called vertices and edges. For an edge e , the vertices of the set I(e) are called the ends of e . The edge e is said to be joining its ends together. An edge e with only one end is called a loop. If two edges have the same set of ends p , then they are called parallel edges or multiple edges. A graph without loops and multiple edges is called a simple graph. The degree (or valence) of a vertex v in a simple graph is the number of edges e which have v as an end. If all vertices have the same degree, then the graph is said to be regular. Two vertices are said to be adjacent iff there exists an edge joining them. The complete graph k_v is a simple graph on v vertices in which any two distinct vertices are adjacent. A path P is an ordered tuple $(v_o , e_1 , v_1 , e_2 , v_2 , \dots , e_n , v_n)$ such that for $i = 1 , \dots n$, v_{i-1} and v_i are the ends of e_i . The path P is said to join the vertices v_o and v_n . The integer n is the length of the path n . The graph is said to be connected iff there exists a path joining any two vertices of the graph.

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The distance between two vertices x and y is the smallest integer n for which there exists a path of length n joining x and y. An incidence structure π is a triple (P , B , I) where P and B are disjoint sets and $I \subseteq P \times B$. The elements of P and B are respectively called points (or treatments) and lines (or blocks). For $p \in P$ and $b \in B$ if $(p , b) \in I$, we say that the point p and the block b are mutually incident and the ordered pair (p , b) is called a flag. The flag graph of π is a simple graph whose vertices are the flags of π and two flags (p , b) and (p' , b') are adjacent iff either $p = p'$, $b \neq b'$ or $p \neq p'$ and $b = b'$. Let v , k , λ be positive integers. A (v , k , λ) - balanced incomplete block design (bibd) is an incidence structure π with v points such that every block is incident with exactly k points and for any two distinct points x and y, there are exactly λ blocks incident with both x and y. If moreover the number of blocks is equal to the number of points, then the b i b d is called a symmetric b i b d.

Association schemes were implicitly considered by Bose and Nair in [5]. Association schemes were explicitly introduced by Bose and Shimamato [6]. Let V be a finite set. A binary symmetric relation on V is a mapping $R : V \times V \to \{0,1\}$ where $R(x , y) = R(y , x)$ $\forall x , y \in V$. Such a relation can be viewed as a $(v \times v)$ - symmetric 0-1 matrix where v is the number of elements of the set V. Let v , m , p^i_{jk}, $i,j,k = 0,1,\ldots m$, be non negative integers. An association scheme $\mathcal{A}$ with parameters $(v, m, p^i_{jk}, i, j, k = 0, 1,\ldots m)$ consists of a finite set V and non null binary symmetric relations $R_0 , R_1 ,\ldots R_m$ on V such that R_0 is the

identity relation and

(i) $\sum_{i=o}^{m} R_i = J$ and

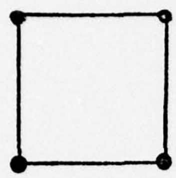
(ii) $\forall\, j , k = 0 , 1 , \ldots m ,\quad R_j R_k = \sum_{i=o}^{m} p_{jk}^{i} R_i$

where $J(x , y) = 1$, for all $x , y \in V$.

Elements of V are called vertices or treatments. If $R_i(x , y) = 1$, then we say that x and y are ith associates, $i = 0 , 1 , \ldots m$. The condition (i) states that for any two vertices x and y , there exists exactly one integer i such that $o \leq i \leq m$ and x and y are ith associates. For two vertices x and y and $j , k = 0 , 1, \ldots m$, let $p_{jk}(x , y)$ denote the number of vertices z such that z and x are jth associates and z and y are kth associates. The matrices $R_1 , \ldots , R_m$ define an edge coloring of K_v , the complete graph on v vertices by m colors. If the vertices x and y are ith associates, then the edge joining them is colored by the ith color , $i = 1 , 2 , \ldots m$. The graph consisting of the edges of the ith color is called the ith associate graph. The parameters of an association scheme are not all independent. For instance for a 2-class scheme, it is sufficient to specify the 4 parameters $v , p_{11}^{o} , p_{11}^{1}$ and p_{11}^{2} . Graphs of the first associates in a 2-class scheme are also called strongly regular graphs. A strongly regular graph with parameters $(v , p_{11}^{o} , p_{11}^{1} , p_{11}^{2})$ contains v vertices such that (1) every vertex is incident with p_{11}^{o} edges, (2) for any two adjacent vertices x and y , there are exactly p_{11}^{1} vertices z which are adjacent to both x and y and (3) for any two nonadjacent vertices x and y there are exactly p_{11}^{2} vertices z which are adjacent to both

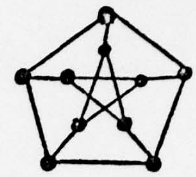
x and y . We give below some examples of strongly regular graphs.

Example 1



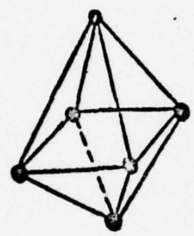
parameters (4, 2, 0, 1)

Example 2



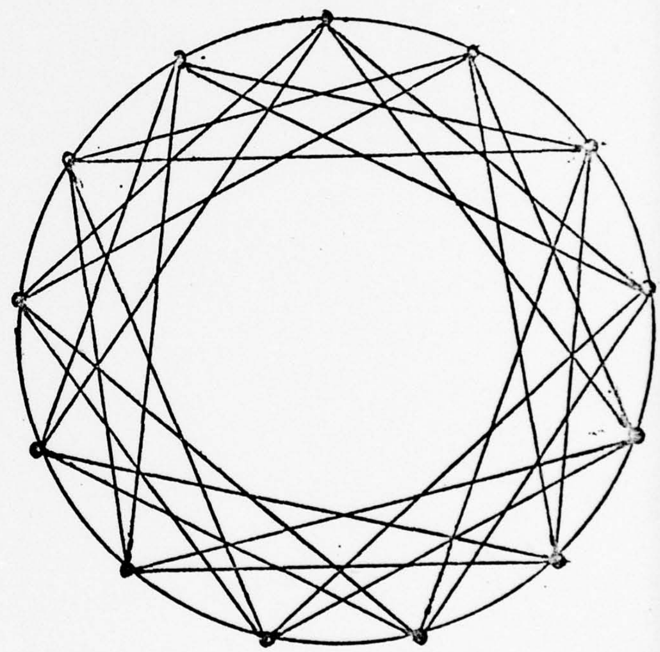
parameters (10, 3, 0, 1)

Example 3



parameters (6, 4, 2, 4)

Example 4



In example 4 vertices are the elements of GF(13) . Two elements x and y are adjacent iff x-y is a square in GF(13) .

Association schemes are readily available in nature. We only describe a few infinite families of association schemes. An exhaustive survey of

2-class schemes (or strongly regular graphs) can be found in [7].

(1) Three class schemes of projective planes. Let π be a finite projective plane of order n. Let $G(\pi)$ be the flag graph of π. We define two vertices to be ith associates iff the distance between them in $G(\pi)$ is i where i is a non negative integer. This definition of the association relations satisfy the properties of a 3-class association scheme $\mathcal{A}(\pi)$ with parameters

$$\begin{aligned} & v = (n+1)(n^2+n+1), && p^0_{11} = 2n, && p^0_{22} = 2n^2, \\ & p^1_{11} = n-1, && p^1_{12} = n, && p^1_{22} = n(n-1), \\ & p^2_{11} = 1, && p^2_{12} = n-1, && p^2_{22} = n, \\ & p^3_{11} = 0, && p^3_{12} = 2 \text{ and } && p^3_{22} = 4(n-1). \end{aligned} \qquad \ldots\ldots(1)$$

The remaining parameters of the scheme can be expressed in terms of the parameters given above. Conversely it can be shown that for any association scheme $\mathcal{A}$ with parameters given in (1), there exists a projective plane π of order n such that $\mathcal{A}$ and $\mathcal{A}(\pi)$ are isomorphic. In other words an association scheme $\mathcal{A}$ with parameters (1) is really nothing but the projective plane π. It is interesting to note that the association scheme $\mathcal{A}$ does not distinguish between points and lines of the projective plane.

(2) Three class schemes of symmetric balanced incomplete block designs (bibd). Let $v' > k > \lambda > o$ be integers. Consider a symmetric b i b d π with parameters (v', k, λ). Let $G(\pi)$ be a bipartite graph whose vertices are the points and lines of π and two vertices adjacent iff one of them is a treatment and the other is

a block incident with the treatment. Two vertices x and y are defined to be ith associates iff the distance between x and y in $G(\pi)$ is i where i is a non negative integer. The association relations so defined produce a 3-class association scheme $\mathcal{A}(\pi)$ with parameters

$$
\begin{array}{llll}
v = 2v' , & p_{11}^{0} = k , & p_{22}^{0} = v' - 1 , & \\
p_{11}^{1} = 0 , & p_{12}^{1} = k - 1 , & p_{22}^{1} = 0 , & \\
p_{11}^{2} = \lambda , & p_{12}^{2} = 0 , & p_{22}^{2} = v' - 2 , & \qquad \ldots\ldots(2) \\
p_{11}^{3} = 0 , & p_{12}^{3} = k & \text{and } p_{22}^{3} = 0 . &
\end{array}
$$

Conversely if $\mathcal{A}$ is a 3-class scheme with parameters given by (2), then there exists a (v', k, λ) - symmetric bibd π such that $\mathcal{A}$ and $\mathcal{A}(\pi)$ are isomorphic.

(3) Association schemes of the projective spaces. Let m and d be positive integers satisfying $m \leq \frac{d}{2}$ and $d \geq 4$. Let π denote the projective space $PG(d - 1, q)$. Construct an association scheme with $(m - 1)$ - flats as the vertices. Two $(m - 1)$ - flats are ith associates iff their intersection is an $(m - 1 - i)$ - flat, $i = 0, 1, \ldots m$. The association relations so defined satisfy the properties of an m-class scheme. This scheme will be denoted by $P(m, q, d)$. This scheme can be described in terms of $G(\pi)$, the graph of the first associates. Two vertices are ith associates iff the distance between them in $G(\pi)$ is i, $i = 0, 1, 2, \ldots m$. The graph of the first associates of $P(2, q, d)$ is also called the line graph of $PG(d - 1, q)$.

(4) Association schemes of the restriction of projective spaces. Let m and d be positive integers satisfying $m \leq \frac{m+d}{2}$. Let π be the projective space $PG(m + d - 1 , q)$. Let Σ_{d-1} be a $(d-1)$ - flat of π. Construct an association scheme whose vertices are the $(m-1)$ - flats, which do not intersect Σ_{d-1}. Two flats are ith associates iff they intersect in an $(m - 1 - i)$ - flat, $i = 0 , 1 , \ldots m$. The association relations so defined satisfy the properties of an m-class scheme. This scheme is denoted by $R(m , q , d)$. $R(m , q , d)$ can be described in another way. Let V be a vector space of dimension d over $GF(q)$. The vertices of $R(m , q , d)$ are m-tuples $(x_1 , x_2 , \ldots, x_m)$ belonging to $\underbrace{V \times V \times \ldots \times V}_{m}$. Two m-tuples $(x_1 , x_2 , \ldots, x_m)$ and $(y_1 , y_2 , \ldots, y_m)$ are ith associates iff the dimension of the subspace spanned by the vectors $x_1-y_1 , \ldots, x_m-y_m$ is i, $i = 0 , 1 , \ldots m$. $R(m , q , d)$ also can be described in terms of its graph of the first associates G. Two vertices are ith associates iff the distance between them in G is i, $i = 0 , 1 , \ldots m$.

2. Study of association schemes.

There had been three kinds of investigations about association schemes; (1) non existence of schemes with certain parameters, (2) construction of schemes and (3) uniqueness of certain schemes.

Bose and Mesner [4] introduced the algebra of the association matrices. Let $\mathcal{Q}$ be an association scheme with association matrices $R_0 , R_1 , \ldots, R_m$ and parameters $(v , m , p^i_{jk} , i , j , k = 0 , 1 , \ldots, m)$. Consider the set of matrices $\sum_{i=0}^{m} c_i R_i$ where c_i's are arbitrarily chosen rational

coefficients. From the defining properties it is easily seen that this set is closed under addition and multiplication. Therefore we get an algebra of matrices called the association algebra. Let $P_k = ((p^j_{ik}))$ be an $(m+1) \times (m+1)$ - matrix whose entry in the ith row and jth column is the parameter p^j_{ik}. It can be seen that the parameter matrices $P_o, P_1, \ldots, P_m$ generate an algebra over the rationals which is isomorphic to the association algebra. Let $R = \sum_{i=o}^{m} c_i R_i$ and $P = \sum_{i=o}^{m} c_i P_i$. The matrices R and P have the same minimum polynomials and the same set of distinct eigen values $\theta_o, \theta_1, \ldots, \theta_u, u \leq m$. Let α_i be the multiplicity of the eigen value θ_i in the matrix R, $i = 0, 1, \ldots u$. For any integer p, the matrix R^p can be expressed as a linear combination $\sum_{i=o}^{m} c_{pi} R_i$ where c_{pi}'s depend on c_i's and the parameters of the scheme. Computing trace R^p in two different ways, we get the equation

$$\sum_{i=o}^{m} \alpha_i \theta_i^p = v c_{po}, \quad p = 0, 1, \ldots, u \qquad \ldots\ldots(3)$$

In the equations (3), all quantities except $\alpha_o, \alpha_1, \ldots, \alpha_u$ can be computed explicitly as functions of the parameters of the scheme. Hence a necessary condition for the existence of a scheme with parameters $(v, m, p^i_{jk}, i, j, k = 0, 1, \ldots m)$ is that the equations (3) have integral solutions for the unknowns $\alpha_o, \alpha_1, \ldots \alpha_u$. This necessary condition is a very strong condition and eliminates many parameter sets. Since the association algebra is commutative, the algebra of the parameter matrices is also commutative. Therefore the parameter matrices commute

pairwise. The commutativity of the parameter matrices also imply several relations among the parameters. For instance some necessary conditions are

$$\sum_{k=0}^{m} p_{jk}^{i} = p_{jj}^{o} \ , \ p_{ii}^{o} \ p_{jk}^{i} = p_{jj}^{o} \ p_{ik}^{j} \quad , \ \forall \ i, j, k = 0, 1, \ldots, m \ . \quad \ldots\ldots(4)$$

The algebra of association matrices had been used successfully to prove the nonexistence of Moore graphs of diameter greater than 2 . The diameter of a connected graph is the largest possible distance between two vertices of the graph. The girth of a graph is the smallest possible number of edges in a polygon of the graph if such a polygon exists. A Moore graph of diameter k and valence k is a graph with valence d , diameter k and girth $(2k + 1)$. In such a graph we can define two vertices to be ith associates iff the distance between them is i , $i = 0 , 1 , \ldots k$. This defines a k-class association scheme. Hoffman and Singleton [16] exploited the association algebra to prove that Moore graphs of valence $d > 2$ and diameter 3 do not exist. Vijayan [23], Bannai and Ito [1] and Damerell [10] used the association algebra to prove the following theorem.

Theorem 1. For $d > 2$, $k \geq 3$, Moore graphs of valence d and girth $(2k + 1)$ do not exist.

3. Uniqueness of association schemes. Study of uniqueness of association schemes was started by Connor [9] in connection with the triangular scheme. Shrikhande [19] did pioneering work in proving the uniqueness of the L_2-scheme and Bruck [8] in a certain sense proved the uniqueness of the L_r-scheme. Bose [2] generalized the methods of these workers and proved an important theorem for partial geometries. Let r , k and t

be positive integers. An (r, k, t) - partial geometry π is an incidence structure of points and lines such that (1) every line is incident with exactly k points, (2) every point is incident with exactly r lines, (3) two distinct points are incident with at most one common line and (4) given a point p and a nonincident line ℓ, there are exactly t lines which are incident with p and also a point of ℓ.

It is easy to see that the dual of an (r, k, t) - partial geometry is a (k, r, t) - partial geometry. For a partial geometry π, we define a simple graph $G(\pi)$ whose vertices are the points of π and two points are adjacent in the graph iff there is a line in π incident with both the points. $G(\pi)$ is a strongly regular graph with parameters

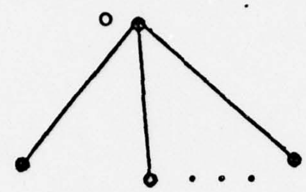
$$v = \frac{k}{t}((r-1)(k-1) + t), \quad p_{11}^{0} = r(k-1),$$
$$p_{11}^{1} = (k-2) + (r-1)(t-1) \quad \text{and} \quad p_{11}^{2} = r\,t \qquad \ldots\ldots(5)$$

$G(\pi)$ is called an (r, k, t) - geometric strongly regular graph. A strongly regular graph with parameters given in (5) is called an (r, k, t) - pseudogeometric graph.

Theorem 2. (Bose [2]). Let r, k and t be positive integers satisfying $k > \frac{1}{2}(r(r-1) + t(r+1)(r^2 - 2r + 2))$ and G be an (r, k, t) - pseudogeometric strongly regular graph. Then there exists a unique (r, k, t) - partial geometry π such that G and $G(\pi)$ are isomorphic.

The concepts of claw and clique play an important role in the proof of Bose's theorem. An s-claw of G is an ordered pair (o, U) where

U is a set of s-vertices such that no two vertices of U are pairwise adjacent and o is a vertex adjacent to all vertices of U.



A clique is a subset of vertices such that any two are pairwise adjacent. Let (o , U) be an $(r-1)$ - claw of G and $f(i)$ be the number of vertices x of G which are adjacent to o and also adjacent to exactly i of the $(r-1)$ - vertices in the set U. Using the parameters, one gets bounds for the power sums $\Sigma\, i^t f(i)$ for $t = 0, 1, 2$ which in turn gives information about the frequencies $f(o)$ and $f(1)$. This method is commonly known as the method of moments. One then proves that G contains no $(r+1)$ - claw and that every pair of adjacent vertices is contained in a maximal clique of size at least $k - (r-1)^2(t-1)$. Such cliques are called grand cliques. After some simple manipulations one proves that (i) these grand cliques have size exactly equal to k, (ii) every vertex is contained in exactly r grand cliques, (iii) every pair of adjacent vertices is contained in a unique grand clique and (iv) given a clique C and a vertex o not in C, exactly t vertices of C are adjacent to o. Hence if one takes the vertices of G to be points and the grand cliques as lines, one easily gets an (r , k , t) - partial geometry π with $G(\pi)$ isomorphic to G.

Recently Bose, Shrikhande and Singhi [7] made an important generalization of theorem 2. Consider an incidence structure (P , B , I) where P and B are disjoint sets and $I \subseteq P \times B$. Elements of P and B are respectively called points and blocks. For two points p and p', let $m(p , p')$

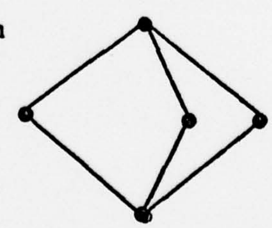
denote the number of blocks b incident with both p and p'. For $p \in P$, $b \in B$, define $n(p, b) = \Sigma\, m(p, p')$ where the sum is over all points p' incident with b. Let r, k, t and c be non negative integers. An incidence structure π of points and blocks is said to be an (r, k, t, c) - partial geometric design iff (1) every block is incident with exactly k points, (2) every point is incident with exactly r blocks, (3) for an incident pair (p, b), $n(p,b) = r+k-1+c$ and (4) for a nonincident pair (p, b), $n(p, b) = t$. It is easy to see that an (r, k, t, o) - partial geometric design is an (r, k, t) - partial geometry. Define a graph $G(\pi)$ whose vertices are the points of π and two points p and p' are joined by $m(p, p')$ distinct edges. The graph $G(\pi)$ is called an (r, k, t, c) - geometric graph and possesses some "regularity properties." An arbitrary graph G with similar "regularity properties" is called an (r, k, t, c) - pseudogeometric graph. Bose, Shrikhande and Singhi proves that if k is greater than a certain function of r, t and c, then every (r, k, t, c) - pseudo-geometric graph is an (r, k, t, c) - geometric graph.

Bose's theorem played an important role in the development of the subject. A theorem of the present author and a theorem of Alan J. Hoffman also played important roles. Let G be a simple graph, i.e. a graph without loops and multiple edges. The line graph $L(G)$ is a graph whose vertex set is the edge set of G. Two vertices e and e' of $L(G)$ are adjacent if the corresponding edges of G have a common incident vertex. For a simple graph G with v vertices, the adjacency matrix

of G is a $(v \times v)$ - matrix $A = ((a_{ij}))$ where $a_{ij} = 1(0)$ iff the ith vertex and the jth vertex are adjacent (not adjacent), $i, j = 1,2,\ldots v$. The eigen values of the adjacency matrix are called the eigen values of the graph.

Theorem 3. (Ray-Chaudhuri [18], Characterization of line graphs). Let G be a finite simple graph such that the number of edges of G is greater than the number of vertices of G. Then the minimum eigen value of $L(G)$ is -2. Conversely, let H be a simple graph with the minimum eigen value equal to -2, the minimum valence not less than 46 and the property that for any two adjacent vertices x and y, there are at least two distinct vertices z and z' adjacent to x and not adjacent to y. Then there exists a simple graph G with $L(G)$ isomorphic to H.

Proof of the first part of the theorem is easy. The proof of the converse part of the theorem uses some interesting ideas. It is easily seen that a line graph $L(G)$ has a class of cliques $\mathcal{C}$ such that every vertex of $L(G)$ is contained in exactly two cliques of $\mathcal{C}$ and every edge is contained in exactly one clique of $\mathcal{C}$. Conversely one can show that if a simple graph H contains such a class of cliques $\mathcal{C}$, then H will be a line graph. To build the class of cliques $\mathcal{C}$ in H, first one shows that H does not contain a 3-claw. Since the minimum eigen value of H is -2, many graphs can not occur as induced subgraphs of G. To give an example, H can not contain the graph



as an induced subgraph.

A graph H is said to be an induced subgraph of G iff $V(H) \subseteq V(G)$ and every edge of G with ends belonging to $V(H)$ is an edge of H. The minimum eigen value of the graph F is smaller than -2. If F were an induced subgraph of H, then by the minimum principle the minimum eigen value of H will be strictly smaller than -2. One builds up a list of inadmissible subgraphs for H and uses these subgraphs to prove the nonexistence of a 3-claw.

Disjoint unions of the complete graph K_v is easily seen to be a strongly regular graph. The class of these graphs and their compliments is called the class of trivial strongly regular graphs. Hoffman proved the following important theorem.

Theorem 4. (Hoffman [13])

Let m be a positive integer. Then there exists a function $f(m)$ such that for every non trivial strongly regular graph G with minimum eigen value $-m$, the parameter p_{11}^2 is not greater than $f(m)$.

Using the theorems stated above, one can prove the following theorem.

Theorem 5.

Let m be a positive integer and G be a non trivial strongly regular graph with minimum eigen value equal to $-m$. Then with finitely many exceptions G is either an (m, k, m) - pseudogeometric strongly regular graph or an $(m, k, m-1)$ - pseudogeometric strongly regular graph where k is a positive integer depending on G.

Using the relations (3), one can see that the three parameters p_{11}^o, p_{11}^1 and p_{11}^2 determine the parameter v for a $(v, p_{11}^o, p_{11}^1, p_{11}^2)$ - strongly regular graph. For a fixed parameter triple $(p_{11}^o, p_{11}^1, p_{11}^2)$,

there are only finitely many nontrivial strongly regular graphs. Using the association algebra, one can show that the eigen values of G are p_{11}^{o} , $-m$ and $p_{11}^{1} - p_{11}^{2} + m$. Also denoting by r the multiplicity of the eigen value $p_{11}^{1} - p_{11}^{2} + m$, we get the following two equations

$$m^2 + (p_{11}^{1} - p_{11}^{2})\, m + p_{11}^{2} = p_{11}^{o} \qquad .....(6)$$

$$p_{11}^{2}\, r\, (p_{11}^{1} - p_{11}^{2} + 2m) = (m - 1)\, p_{11}^{o}\, (m\, p_{11}^{1} - (m - 1)\, p_{11}^{2} + m(m + 1))$$

Given m , p_{11}^{2} and p_{11}^{1} , the equation (6) determines p_{11}^{o} . Therefore there are finitely many nontrivial strongly regular graphs with fixed values for the minimum eigen value $-m$ and the parameters p_{11}^{1} and p_{11}^{2} . By Hoffman's theorem, the parameter p_{11}^{2} of G will be bounded by a function $f(m)$. Hence we need to consider only finitely many values for the parameter p_{11}^{2} . Suppose we fix m and p_{11}^{2} and try to find the possible values for p_{11}^{1} . Let x be the unknown value of p_{11}^{1} . From the equations (6) , we can derive the equation

$$p_{11}^{2}\, r\, (x - p_{11}^{2} + 2m) = (m - 1)(m\, x - (m - 1)\, p_{11}^{2} + m^2)\, x$$
$$(m\, x - (m - 1)\, p_{11}^{2} + m(m + 1) \qquad .....(7)$$

We use the fact that r is a positive integer. If $p_{11}^{2} \neq m^2$ or $m(m - 1)$, the equation (7) leads to finitely many possible values for p_{11}^{1} . Hence if $p_{11}^{2} \neq m^2$ or $m(m - 1)$, there are finitely many nontrivial strongly

regular graphs with fixed values for the parameters $-m$ and p_{11}^2. Suppose $p_{11}^2 = m^2$. If we let $k = p_{11}^1 + 2 - (m-1)^2$, then a strongly regular graph with parameters $(m, p_{11}^2 = m^2, p_{11}^1)$ is an (m, k, m) - pseudogeometric strongly regular graph. Similarly if $p_{11}^2 = m(m-1)$, we let $k = p_{11}^1 + 2 - (m-1)(m-2)$. A non trivial strongly regular graph with parameters $(m, p_{11}^2 = m(m-1), p_{11}^1)$ is then an $(m, k, m-1)$ - pseudogeometric strongly regular graph.

C. C. Sims [20] proved a beautiful characterization theorem for a large class of strongly regular graphs. We define 4 *classes of strongly regular* graphs as follows.

(1) Line graph of the complete graph on n vertices where n is a positive integer. This graph is a $(2, n-1, 2)$ - geometric strongly regular graph with parameters $v = \binom{n}{2}$, $p_{11}^0 = 2(n-2)$, $p_{11}^1 = n-2$ and $p_{11}^2 = 4$.

(2) Line graph of the complete bipartite graph on $n + n$ vertices where n is a positive integer. This graph is a $(2, n, 1)$ - geometric strongly regular graph with parameters $v = n^2$, $p_{11}^0 = 2(n-1)$, $p_{11}^1 = n-2$ and $p_{11}^2 = 2$.

(3) Line graph of the projective space $PG(d-1, q)$ where q is a prime and d is a positive integer not less than 4. This is the graph of the first associates of the $P(2, q, d)$ - scheme. This graph is a $(q+1, k, q+1)$ - geometric strongly regular graph with $k = \frac{q^{d-1} - 1}{q-1}$.

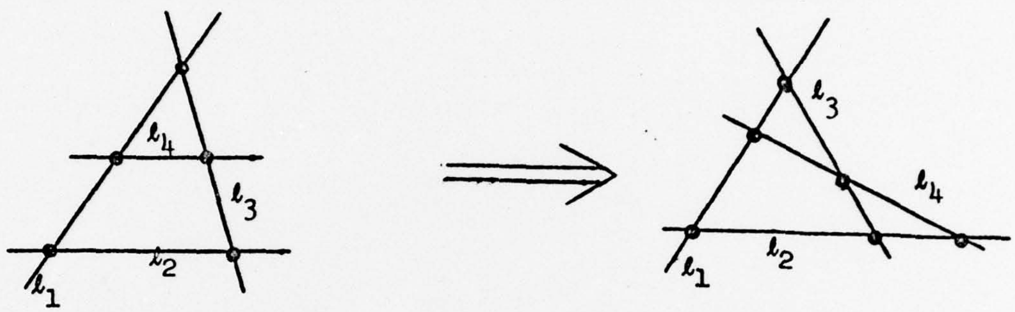
(4) Graph of the first associates of the $R(2, q, d)$ - scheme where q is a prime power and d is a positive integer not less than 2. This graph is a $(q+1, k, q)$ - geometric strongly regular graph with $k = q^{d-1}$.

All the graphs defined above satisfy the 4-vertex condition. Let x and y be two vertices of a strongly regular graph G. Let S_{xy} be the set of vertices z which are adjacent to both x and y. Let $\alpha(x,y)$ denote the number of edges of G with both ends belonging to S_{xy}. The 4-vertex condition requires that there exists two numbers α_1 and α_2 such that if the distance between x and y is i, then $\alpha(x,y) = \alpha_i$, $i = 1,2$. Actually the graphs of the 4-classes defined above have rank 3 automorphism groups.

Theorem 6. (C. C. Sims [20]). Let m be a positive integer greater than 1. Then there exists a finite class of graphs $\mathcal{S}$ such that if G is a non trivial strongly regular graph with minimum eigen value equal to $-m$ and satisfying the 4-vertex condition, then either $G \in \mathcal{S}$ or G belongs to one of the 4-classes of graphs defined above.

Sims proved his theorem under the assumption that G has a rank 3 automorphism group. Higman [12] and also some other workers observed that Sims' proof remains valid under the weaker 4-vertex condition. The proof of Sims' theorem depends on the previous theorems. For the case $m = 2$, the proof comes out fairly quickly as an application of theorem 3. Using the theorem 2 and 3, one can see that G can be assumed to be either an (m, k, m) - geometric strongly regular graph or an $(m, k, m-1)$ - geometric strongly regular graph where k is some positive integer. Suppose $G = G(\pi)$ where π is an (m, k, m) - partial geometry. Then

π^* the dual of π is a (k , m , m) - partial geometry. The 4-vertex condition for G is used to show that π^* satisfies the Pasche axiom, i.e. if ℓ_1 , ℓ_2 , ℓ_3 and ℓ_4 are lines such that no three have a common point and 5 pairs of lines intersect, then the 6th pair also must intersect. The condition can be described by the following picture.



It is then easily seen that π^* is a projective space. If $G = G(\pi)$ where π is an $(m , k , m-1)$ - partial geometry, then also π^* satisfies the Pasche axiom. Sims then essentially shows that π^* is isomorphic to an incidence structure (P , L , I) where P is the set of points of $\Sigma_n - \Sigma_{n-2}$, Σ_n is a projective space of dimension n , Σ_{n-2} is an $(n-2)$ - flat of Σ_n , L is the set of lines of Σ which do not intersect Σ_{n-2} , incidence relation is that in Σ_n and n is a suitable positive integer. This result shows that if we assume the Pasche axiom, then the projective space Σ_n for $n \geq 3$ can be reconstructed from the lines which do not intersect a distinguished $(n-2)$ - flat Σ_{n-2} . This result itself is very interesting. Unfortunately the proof given by Sims is very long. For the case $m = 2$, the 4-vertex condition is not necessary and one can prove the following theorem.

Theorem 7. There exists a finite class of graphs $\mathcal{J}$ such that if G is

a strongly regular graph with -2 as the minimum eigen value, then either $G \in \mathcal{S}$ or G is a line graph of a complete graph or G is a line graph of a complete bipartite graph.

Let G be a finite connected graph. For two vertices x and y, and integers j and k, $p_{jk}(x,y)$ will denote the number of vertices z which have distance j from x and distance k from y. If for all pairs of vertices (x,y) with distance i, the numbers $p_{jk}(x,y)$'s are equal, the common value is denoted by p^i_{jk} and we say that the distance parameter p^i_{jk} exists.

Let n and m be positive integers satisfying the inequality $n \geq 2m$ and X be an n-set. We can define an association scheme with the m-subsets of X as the vertices. Two m element subsets Y and Y' are defined to be ith associates iff their intersection contains exactly $m - i$ elements, $i = 0, 1, \ldots m$. Dowling [11] denotes the graph of the first associates of this scheme by G^n_m. Dowling proves that the graph G^n_m (or the corresponding association scheme) can be reconstructed from a few of its properties.

<u>Theorem 8</u>. (Dowling [11]) Let n and m be positive integers such that $n > 2m(m-1) + 1$. Let G be a connected graph on $\binom{n}{m}$ vertices with distance parameters $p^0_{11} = m(n-m)$, $p^1_{11} = n-2$ and $p_{11}(x,y) \leq 4$ for all pairs of vertices (x,y) with distance more than 1. Then G is isomorphic to G^n_m.

Alan Sprague and the author in a certain sense generalized the results of Sims. Our results are about reconstruction of the $P(m, q, d)$ and $R(m, q, d)$ - schemes for $m \geq 3$. We can prove that if d is large compared to m and q, then such reconstruction is possible. We do not

need to assume all the parameters of the scheme. The existence and correct values for the parameters in the P_1 - matrix are sufficient for the purpose of reconstruction of these schemes. The investigation is not complete yet. We think we will be able to reduce the number of parameters considerably. Also we need only assume the existence of the graph of the first associates with certain properties. For the case $m = 3$, we have the following theorems.

Theorem 9. (Alan Sprague and D. K. Ray-Chaudhuri [21], Uniqueness of the $P(3, q, d)$ - schemes.)

Let q and d be integers satisfying $q \geq 2$, $d \geq 9$ and $(q, d) \neq (2, 9)$. Let $k = \dfrac{q^d - q^2}{q^3 - q^2}$. Let G be a finite simple connected graph with distance parameters

$$p_{11}^{o} = (q^2+q+1)(k-1) , \; p_{11}^{1} = (k-2) + q^2(q+1) , \; p_{11}^{2} = (q+1)^2 \text{ and } p_{31}^{2} = q^2(k-q^2-q-1) .$$

Then q is a prime power and G is isomorphic to the graph of the first associates of $P(3, q, d)$.

Theorem 10. (Alan Sprague and D. K. Ray-Chaudhuri [22], Uniqueness of the $R(3, q, d)$ - schemes.)

Let q and d be integers satisfying $q \leq 2$, $d \geq 6$, $(q, d) \neq (2, 6)$. Let G be a finite simple connected graph which satisfies the 4-vertex condition and has the distance parameters $p_{11}^{o} = (q^2+q+1)(q^d-1)$, $p_{11}^{1} = q^d+q^3-q-2$, $p_{11}^{2} = q^2+q$, $p_{31}^{2} = q^{d+2} - q^4$ and $p_{21}^{3} = q^2(q^2+q+1)$. Then G is isomorphic to the graph of the first associates of the $R(3, q, d)$ - scheme and q is a prime power.

Theorem 8 and Theorem 9 are deep theorems and their proofs are

unfortunately very long. I try to give below an idea about the structure of the proof. Let π and $G(\pi)$ respectively denote the projective space $PG(d-1\ ,\ q)$ and the graph of the first associates of the $P(3,q,d)$ - scheme. Then we have the following correspondences.

Correspondence between objects in π and $G(\pi)$

π	Class of associated flats of π	$G\ (\pi)$
plane 		vertex
line 	 the pencil of planes containing the line	A clique of size $\dfrac{q^{d-2}-1}{q-1}$
. point	 the pencil of lines containing the point	A class of cliques containing $\dfrac{q^{d-1}-1}{q-1}$ cliques such that any two cliques of the class have exactly one common vertex.

Given a graph G with the properties stated in theorem 9, we first prove the existence of large cliques. We prove the existence of a family of cliques of size $\dfrac{q^{d-2}-1}{q-1}$ such that every edge of the graph is contained in exactly one of these "grand cliques". Next one proves the existence of complexes of cliques. A complex of cliques is a class of $\dfrac{q^{d-1}-1}{q-1}$ cliques such that any two cliques of the class have exactly one common vertex. Complexes and cliques will respectively correspond

to points and lines of the projective space. One proves that the incidence structure of complexes and cliques satisfies the axioms of the projective space. One of the interesting by-product of theorem 9 is the fact that the projective space can be reconstructed from the incidence structure of the lines and the planes. One needs to assume very few properties of this incidence structure for this reconstruction problem. The proof of theorem 10 is much more difficult than that of theorem 9. In fact this is not surprising. It can be easily seen that the $R(3,q,d)$ - scheme in a certain sense is a sub-scheme of the $P(3,q,d+3)$ - scheme. Theorem 10 also produces a reconstruction theorem for projective spaces. Let Σ_{d+2} be a finite projective space of dimension $(d+2)$ and Σ_{d-1} be a $(d-1)$ flat. Let π be the incidence structure of the lines and planes of Σ_{d+2} which do not intersect Σ_{d-1}. It can be shown that a few properties of the incidence structure π are sufficient for the reconstruction of the projective space.

In the proofs of theorems 9 and 10, to prove the existence of large cliques we apply the Bose-Laskar theorem. The theorem of Bose and Laskar shows that if in a graph G the parameters p_{11}^{o} and p_{11}^{1} exist and the remaining numbers $p_{11}(x,y)$ are not too large, then one can prove the existence of a nice family of large cliques.

<u>Theorem 11</u>. (Bose and Laskar [3])

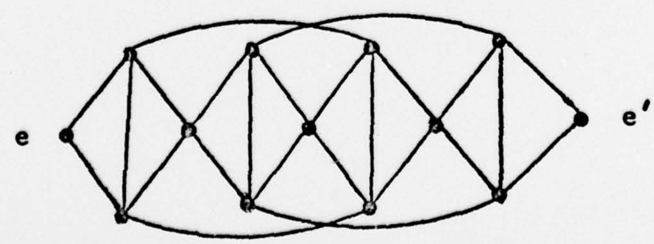
Let $r \geq 1$, $k \geq 2$, $e \geq o$ and $b \geq o$ be integers such that $k > \max(1 + b + e(2r - 1)\ ,\ 1 + \frac{1}{2}(r + 1)(rb - 2e))$. Let G be a graph such that the distance parameters p_{11}^{o} and p_{11}^{1} are given by $p_{11}^{o} = r(k-1)$, $p_{11}^{1} = k - 2 + e$ and $p_{11}(x,y) \leq 1 + b$ for pairs of vertices (x,y) with

distance greater than 1 . Then every vertex is contained in exactly r grand cliques and every edge is contained in exactly one grand clique where a maximal clique containing at least $k - (r-1)e$ vertices is called a grand clique.

4. Near association schemes. Frequently we come across situations where the symmetric relations defined on a set satisfy many but not all of the properties required of an association scheme. Such structures could be called "near association schemes". A few uniqueness theorems have been proved about such "near association schemes". I give only two illustrations of such theorems. Let $v > k > \lambda > o$ be integers and π be a symmetric (v, k, λ) - bibd . Let $G(\pi)$ denote the flag graph of π. The distance relations in the graph $G(\pi)$ satisfy many but not all the properties required of an association scheme. A theorem of Hoffman and the present author shows that the graph $G(\pi)$ can be reconstructed from its distinct eigen values and connectedness.

Theorem 12. (Hoffman and D. K. Ray-Chaudhuri [15])

Let $v > k > \lambda > o$ be integers and $(v, k, \lambda) \neq (4, 3, 2)$. Let G be a regular simple connected graph with distinct eigen values $2k - 2$, -2 , $k - 2 + \sqrt{k - \lambda}$ and $k - 2 - \sqrt{k - \lambda}$. Then there exists a symmetric bibd π such that G is isomorphic to $G(\pi)$. Further, if $(v, k, \lambda) = (4, 3, 2)$ and G is not isomorphic to the graph



then the statement of the theorem holds. In the diagram e and e' represent the same vertex.

The distance relations in the flag graph of a finite affine plane also satisfy many but not all the properties required of an association scheme. A theorem of Hoffman and the present author proves that the flag graph of an affine plane can be reconstructed from its distinct eigen values and connectedness.

<u>Theorem 13.</u> (Hoffman and D. K. Ray-Chaudhuri [14])

Let n be a positive integer and G be a regular connected simple graph with distinct eigen values $2n - 1$, -2, $\frac{1}{2}(2n - 3 + \sqrt{4n + 1})$, $\frac{1}{2}(2n - 3 - \sqrt{4n + 1})$ and $n - 2$. Then there exists a finite affine plane π such that G is isomorphic to the flag graph of π.

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